Journal of Engineering Mathematics 21: 201–226 (1987)

An unsteady lifting-line theory

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Received 2 March 1987; accepted in revised form 11 May 1987

Abstract. A lifting-line theory is developed for wings of large aspect ratio undergoing time-harmonic oscillations, uniformly from high to low frequencies. The method of matched asymptotic expansions is used to enforce the compatibility of two approximate solutions valid far from and near the wing surface. The far-field velocity potential is expressed as a distribution of normal dipoles on the wake, and its expansion near the wing span leads to an expression for the oscillatory downwash. The near-field flow is two-dimensional. A particular solution is obtained from strip theory and a homogeneous component is added to account for the spanwise hydrodynamic interactions. The compatibility of the inner and outer solutions leads to an integral equation for the distribution of circulation along the wing span. In the zero-frequency limit it reduces to that in Prandtl's lifting-line theory, and for high frequencies it tends to the two-dimensional strip theory. Lift computations are presented for an elliptic and a rectangular wing of aspect ratio $A = 4$.

1. Introduction

The method of matched asymptotic expansions has in recent years been of valuable assistance to fluid dynamicists. Prandtl's boundary-layer and lifting-line theories, and the slender-body theory of incompressible aerodynamics pioneered the way to numerous diverse developments of the method. A systematic presentation is given by Van Dyke [16].

The success of the steady-state lifting-line theory motivated extensions that include unsteady effects. The assumption of a harmonic time-dependence, while not restrictive in a linear problem, introduces an additional length scale associated with the wake wavelength. Many studies have carried out an analysis of the unsteady lifting problem by asymptotically relating the wavelength to either its span or its chord. This approach permits the formal development of leading and higher-order solutions in inverse powers of the aspect ratio A, but prevents their uniform validity for wavelengths ranging from the wing chord to the infinite wavelength limit. James [6] and Van Holten [17] studied the low-frequency range corresponding to wavelengths comparable to the span. The same frequency regime was recently studied by Ahmadi and Widnall [2] correcting the induced downwash derived by James and generalizing its definition. An extension for wings with a swept planform has been derived by Cheng [3] and Cheng and Murillo [4]. The complementary high-frequency limit corresponding to wavelengths comparable to the wing chord has been studied by Guiraud and Slama [5].

In the related area of surface-wave radiation by ships, Newman [10] developed a slender-ship theory valid uniformly from wavelengths comparable to the ship beam, corresponding to the two-dimensional strip theory, to the infinite-wavelength limit which coincides with the slender-body theory of incompressible aerodynamics. It is known in ship hydrodynamics as unified theory. Computations of the hydrodynamic forces on a ship have been carried out by Newman and Sclavounos [11] and were found to be in very good agreement with three-dimensional numerical solutions over a wide range of frequencies. An extension of the theory to surface-wave scattering by a slender ship was derived by Sclavounos [13] with similar success. The unsteady lifting problem is here studied within the same framework.

A theory is developed for the unsteady time-harmonic lifting flow past a high-aspect-ratio wing in steady forward motion. Five frequency ranges have been identified by Cheng [3], corresponding to different ratios of the wing span $2d$ and chord $2l$ to the wake wavelength $\lambda = 2\pi U/\omega$, where U is the wing forward velocity and ω the frequency of oscillation. For a wing of high aspect ratio A, $d/l = O(A)$:

I) $d/\lambda = o(1)$: very low frequency,

II) $d/\lambda = O(1)$: low frequency,

III) $O(1) < d/\lambda < O(A)$: intermediate frequency,

IV) $d/\lambda = O(A)$, $\lambda/l = O(1)$: high frequency,

V) $d/\lambda > O(A)$: very high frequency.

A theory is derived which accounts for three-dimensional spanwise hydrodynamic interactions up to and including terms of order $1/A$ relative to the leading-order two-dimensional solution, uniformly from domain I to domain IV. Prandtl's lifting-line theory is recovered in the infinite-wavelength limit and the two-dimensional strip theory is obtained for wavelengths comparable to the wing chord.

In the spirit of matched asymptotic expansions, the flow regions far from and near the wing surface are examined separately. At distances from the wing axis large compared to its span, the "outer" solution can be approximated by a distribution of concentrated bound vorticity along the span, supplemented by the vorticity in its wake. In light of the equivalence of the vorticity and dipole representations in potential flows, the far-field solution is expressed by a distribution of normal dipoles on the wake which extends from the wing axis to the downstream infinity. Linear theory and the continuity of pressure accross the wake suggest that the wake vorticity is convected by the steady inflow velocity. This condition translates into a sinusoidal variation of the moment of the equivalent dipole distribution, with wavelength λ. For high-aspect-ratio wings, the spanwise variation of the dipole moment is assumed to be gradual. A two-term expansion of the far-field solution is obtained at small radial distances from the wing axis. The leading-order term is a two-dimensional point vortex with strength equal to the oscillatory circulation at each section along the wing span. The next-order term is the anticipated unsteady downwash which represents the leading-order correction for effects of finite span. It is of order $1/A$ relative to the leading-order term, where A is the aspect ratio. Its magnitude vanishes at high frequencies, and tends to that in Prandtl's lifting-line theory in the zero-frequency limit. The spanwise distributions of circulation and downwash are a priori unknown and will be determined from the matching with the "inner" solution.

Near the wing surface the flow is assumed to be two-dimensional. The solution is written as the sum of the two-dimensional strip-theory component, due to the heave or pitch oscillation of each wing section along its span, supplemented by a homogeneous component of unknown strength.

The magnitude of the distribution of circulation and induced downwash in the outer, and of the homogeneous component in the inner problems respectively are determined by enforcing their compatibility in an appropriately defined overlap region. The result is an integral equation of the Cauchy type for the spanwise distribution of circulation. Its kernel depends on the reduced frequency $\omega d/U$, where d is half the wing span, the forcing term is supplied from the two-dimensional strip theory solution and is valid uniformly from high to low frequencies. In the high-frequency limit the kernel becomes exponentially small and strip theory is obtained. In the opposite zero-frequency limit the Prandtl lifting-line integral equation is recovered.

An error analysis has been carried out which established the existence of an overlap matching region over which the errors in the respective expansions of the outer and inner solutions are small in the asymptotic limit $A \to \infty$. The leading-order finite-aspect-ratio corrections are of order $1/A$, uniformly in the frequency range, in agreement with Van Dyke [16, eq. (9.18)] for the steady-state problem. Previous low-frequency lifting-line theories point out the existence of correction terms proportional to $(1/A) \ln A$. It is shown in Section 5 that, due to the uniform validity of the present theory, the appropriate gauge functions are frequency dependent, with terms proportional to $\ln A$ present to order $(1/A^2) \ln A$. Van Dyke [16] also emphasizes that a formal matched asymptotic analysis does not lead to the solution of an integral equation (as in Prandtl's lifting-line theory), producing explicit integrals for the finite-aspect-ratio correction terms. While accepting Van Dyke's point, the solution of an integral equation of the Prandtl type is selected here, motivated by the success of the analogous procedure in the slender-ship problem. The increase in the computational effort is small, and is associated with the solution of a complex linear system of equations with about ten unknowns.

The numerical solution of the integral equation has been obtained using a spectral method. The unknown circulation distribution is approximated by a sine series, and the integral equation is satisfied at a set of cosine-spaced collocation points. Computations of the lift on a rectangular and an elliptic wing of zero thickness and aspect ratio $A = 4$ forced in a time-harmonic heave oscillation, have been found in satisfactory agreement with a numerical lifting-surface solution by Lee [8] based on the vortex-lattice method.

2. Problem definition

A Cartesian coordinate system (x, y, z) and a thin unswept wing fixed relative to it and symmetric with respect to the $y = 0$ plane, are defined in Fig. 1. An incident uniform stream of velocity U flows in the negative x-direction, past the wing which is assumed to undergo time-harmonic heave and/or pitch oscillations around its mean position at a frequency ω and amplitude which may be variable along its span. The fluid is assumed to be incompressible, with density ϱ, and the flow irrotational. The wake is taken to be of infinitesimal thickness and is modelled as a shear layer across which the pressure and normal velocity are continuous, but the tangential velocity vector can sustain a finite jump.

Linear theory requires that the wing thickness is small relative to its chord and the normal velocity on its surface due to its mean angle of attack and vertical oscillatory motion are small relative to the inflow velocity U. We decompose the thickness and lifting problems as

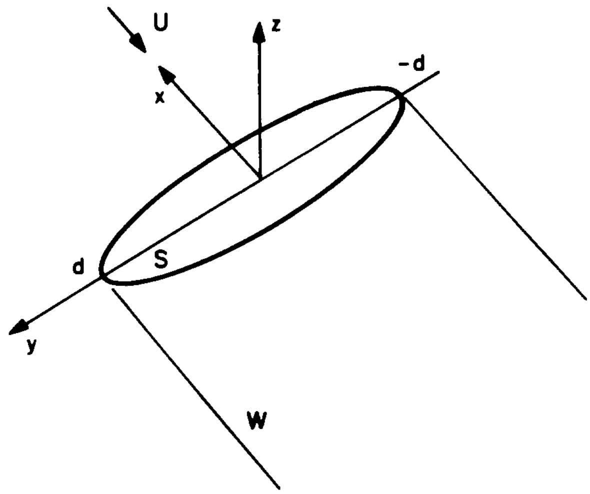

Fig. 1. Problem definition.

in the two-dimensional theory, concentrate on the latter and impose the boundary conditions on the mean wing and wake positions, assumed to coincide with the $z = 0$ plane.

A complex notation is adopted, and is hereafter understood that the real part of the product of all complex quantities with the time factor $e^{i\omega t}$ applies.

The disturbance flow velocity can be expressed as the gradient of a velocity potential ϕ which satisfies the Laplace equation,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \tag{2.1}$$

in the fluid domain, and is subject to the condition

$$\nabla\phi \to 0 \tag{2.2}$$

at infinity, except near the wake. On the mean position of the wing surface S, it satisfies the boundary condition

$$\phi_z = \left(i\omega - U\frac{\partial}{\partial x}\right)\zeta_w(x, y), \quad \zeta_w(x, y) = \xi_3(y) + x\xi_5(y), \tag{2.3}$$

where $\zeta_w(x, y)$ is the wing displacement and $\xi_3(y)$ and $\xi_5(y)$ are the independent complex amplitudes of the heave and pitch oscillatory motions, respectively, which may be variable along the span.

Across the wake the normal-velocity and pressure jumps are zero. The linearized form of the last condition in terms of the jump of the velocity potential is

$$\left.\frac{\partial\phi}{\partial t} - U\frac{\partial\phi}{\partial x}\right|_-^+ = \left(i\omega - U\frac{\partial}{\partial x}\right)\Delta\phi = 0. \tag{2.4}$$

Variation of $\Delta\phi$ in the y-direction is not prevented by (2.4). Condition (2.4) introduces a dynamic length-scale which will hereafter be referred to as wake-wavelength, and is defined by

$$\lambda = \frac{2\pi U}{\omega}. \tag{2.5}$$

The statement of the problem needs to be supplemented by the Kutta condition that the flow velocity is finite at the trailing edge at all times.

The solution of equations (2.1)–(2.4) can be carried out numerically by employing a dipole-distribution or a vortex-lattice method. Popular is its treatment as an initial-value problem where the wing forward and oscillatory motions start from rest. An advantage of this approach is that, for finite time, the wake is of finite length, and there is no need to evaluate the velocity induced by a semi-infinite wake strip. This numerical task is, however, considered neither easy nor inexpensive if high accuracy is required over a wide frequency range.

In the present paper the solution of the same set of equations is attempted using the techniques of matched asymptotic expansions, for wings of high aspect ratio A, defined by

$$A = \frac{4d^2}{S}, \tag{2.6}$$

where S is the area of the wing planform. The solution will be developed uniformly for wake-wavelengths λ ranging from the wing chord to infinity. The outer solution is developed in Section 3, and the inner solution in Section 4. In Section 5 the matching is carried out and the asymptotic errors in the expansions of the outer and inner solutions are studied. It is shown that an overlap region exists over which both are small in the limit $A \to \infty$. The derivation and an analysis of the order of magnitude of the relevant gauge functions over a wide frequency range is carried out in Section 6. The numerical solution of the resulting integral equation for the circulation distribution along the span is outlined in Section 7, where comparisons are made with a numerical lifting-surface solution based on the vortex-lattice method.

3. Outer solution

At distances from the wing axis large compared to its span the flow is not sensitive to the details of the wing geometry. Thus the disturbance due to it can be approximated by a distribution of concentrated oscillatory bound vorticity on the y-axis. This must be supplemented by a trailing vortex sheet with its spanwise vorticity convected by the inflow velocity U, by virtue of (2.4). Moreover, a gradual wing-geometry and normal-velocity variation in the spanwise direction suggest a similar behavior for the trailing streamwise vorticity, by virtue of Kelvin's theorem.

The analysis of the outer solution is facilitated if the distribution of vorticity is replaced by an equivalent distribution of normal dipoles on the wake alone. In two dimensions it is known that a distribution of normal dipoles of variable moment on a semi-infinite straight

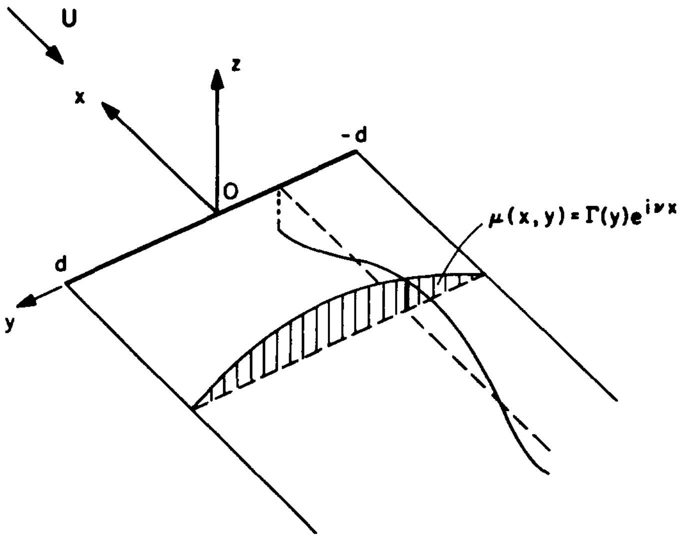

Fig. 2. Outer domain.

line is identical to a distribution of vortices with strength equal to the derivative of the dipole-moment, plus a point vortex at its origin. The proof is based on an integration by parts. The generalization of this result for the streamwise vorticity in the wake of Prandtl's lifting-line theory is derived in Robinson and Laurmann [12]. An attractive property of normal dipoles is that the velocity potential is represented in terms of a scalar singularity distribution, whose strength must be simply twice continuously differentiable to ensure the validity of Kelvin's theorem on the wake (Fig. 2).

The outer disturbance velocity potential is here represented by a distribution of dipoles on the mean position of the wake

$$\phi(x, y, z) = -\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}z}\int_{-d}^{d}\mathrm{d}\eta\int_{-\infty}^{0}\mathrm{d}\xi\,\frac{\mu(\xi, \eta)}{[(x-\xi)^2 + z^2 + (y-\eta)^2]^{1/2}}. \tag{3.1}$$

It is a well-known result in potential flows that the jump of the velocity potential across a plane dipole distribution is equal to the dipole moment, or $\phi^+ - \phi^- = \mu(x, y)$. Using this result, equation (2.4) and the assumption of linearity, the dependence of the dipole moment $\mu(x, y)$ on the x and y coordinates can be cast in the form

$$\mu(x, y) = \mathrm{e}^{i\nu x}\Gamma(y), \tag{3.2}$$

where $\Gamma(y) = 0$ for $|y| > d$, and $\nu = \omega/U$.

Using (3.2), we may rewrite (3.1) as follows:

$$\phi(x, y, z) = -\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}z}\int_{-d}^{d}\mathrm{d}\eta\Gamma(\eta)S(x, y-\eta, z), \tag{3.3}$$

where $S(x, y, z)$ is the velocity potential at the field point (x, y, z) due to a line distribution of Rankine sources of strength $-4\pi\mathrm{e}^{i\nu\xi}$, extending from the origin of the coordinate system to $-\infty$.

We are interested to obtain an expansion of the velocity potential for small values of the radius $r = (x^2 + z^2)^{1/2}$. This expansion is carried out for the Fourier transform of the velocity potential $\phi(x, y, z)$ with respect to the y-coordinate, and the result is then inverted back to the physical y-space. The ensuing analysis does not restrict the magnitude of the wake wavelength relative to the wing span $2d$ which is the only geometrical length scale present in the outer problem.

Define

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{ikx} f(x). \tag{3.4}$$

Taking the Fourier transform of both sides of equation (3.3) and using the convolution theorem, we obtain

$$\tilde{\phi}(x, z; k) = -\frac{1}{4\pi} \tilde{\Gamma}(k) \frac{\mathrm{d}}{\mathrm{d}z} \tilde{S}(x, z; k), \tag{3.5}$$

$$\begin{aligned} \tilde{S}(x, z; k) &= \int_{-\infty}^{0} \mathrm{d}\xi \, \mathrm{e}^{i\nu\xi} \int_{-\infty}^{\infty} \mathrm{d}\eta \frac{\mathrm{e}^{ik\eta}}{[(x - \xi)^2 + z^2 + \eta^2]^{1/2}} \\ &= 2 \int_{-\infty}^{0} \mathrm{d}\xi \, \mathrm{e}^{i\nu\xi} K_0 \{|k|[(x - \xi)^2 + z^2]^{1/2}\}, \end{aligned} \tag{3.6}$$

where $K_0(x)$ is the modified Bessel function of the second kind and order zero, defined in Abramowitz and Stegun [1].

Let $x = r \cos\theta$ and $z = r \sin\theta$. An ascending-series expansion of $S(r, \theta; k)$ is first obtained on the positive x-axis, and is next analytically continued for non-zero values of θ. This technique is based on the observation that $\tilde{S}$ is a solution of the modified Helmholtz equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - k^2\right) \tilde{S}(x, z; k) = 0, \tag{3.7}$$

that it is symmetric with respect to $\theta = 0$ and discontinuous across $\theta = \pi$. Key in the derivation of the ascending-series expansion was the use of the "continuous" addition theorem for modified Bessel functions of the second kind,

$$K_0(x + y) = \frac{2}{\pi} \int_0^{\infty} \mathrm{d}\tau \, K_{i\tau}(x) K_{i\tau}(y), \tag{3.8}$$

where the order of the Bessel functions in the integrand is purely imaginary. The identity (3.8) is attributed to Lebedev and is derived in Sneddon [14]. Its use in (3.6) separates the x- from the ξ-dependence in the argument of K_0, and permits the derivation of an ascending-series expansion in x, which is equal to r on the positive x-axis. This derivation is presented in Appendix 1. For non-zero values of θ, the result is

$$\begin{aligned} \tilde{S}(r, \theta; k) = \frac{2}{|k|} \Bigg\{ &\frac{\gamma^*}{\sinh \gamma^*} \left[I_0(|k|r) + 2 \sum_{m=1}^{\infty} (-)^m \cosh m\gamma \, I_m(|k|r) \cos m\theta \right] \\ &+ \frac{2}{\sinh \gamma} \sum_{m=1}^{\infty} (-)^{m-1} \sinh m\gamma \frac{\partial}{\partial \nu} [I_\nu(|k|r) \cos \nu\theta]_{\nu=m} \Bigg\} \end{aligned} \tag{3.9}$$

where γ^* is the complex conjugate of

$$\gamma = \cosh^{-1}(-i\nu/|k|), \tag{3.10}$$

and $I_m(x)$, $m = 0, \ldots$, are the modified Bessel functions of the first kind and order m, defined in Abramowitz and Stegun [1]. Equation (3.9) is an identity. The small-r expansion of its z-derivative in (3.5) is obtained by formally substituting in (3.9) the ascending-series expansion of the Bessel functions $I_m(x)$ and $[\partial I_\nu(x)/\partial\nu]_{\nu=m}$, and differentiating with respect to the z-coordinate. It follows that

$$\frac{\partial \tilde{S}}{\partial z} = -2\theta + \pi\nu z + i\nu\{-2z(\gamma - 1 + \ln \nu r) - 2\theta x\}$$

$$+ 2|k|z\, L\left(\frac{\nu}{|k|}\right) + O(k^2 r^2 \ln r, \nu^2 r^2 \ln r), \tag{3.11}$$

where only the last term depends on the Fourier wavenumber k. The function $L(w)$ is defined by

$$L(w) = \frac{\cosh^{-1}(iw)}{[(iw)^2 - 1]^{1/2}}(1 + w^2) + iw \ln(2w) - \frac{\pi}{2}w, \tag{3.12}$$

and takes the values $\pi/2$ and 0, for $z = 0$ and ∞ respectively.

Following Newman [10], it is observed that the $k = 0$ limit of the z-derivative of function (3.9) satisfies the two-dimensional Laplace equation with respect to the x- and z-coordinates. It follows from the definition of $\tilde{S}$ in (3.6) that this is the velocity potential at the field point (x, z) due to distribution of normal dipoles with oscillatory moment along the negative x-axis. Taking the z-derivative of the function defined in (3.6) and letting k tend to zero, we obtain

$$D_{2D}(x, z) = \frac{z}{2\pi}\int_{-\infty}^{0} d\xi \frac{e^{i\nu\xi}}{z^2 + (x - \xi)^2}. \tag{3.13}$$

The equivalence of normal dipoles to vortices in two dimensions follows by substituting in (3.13) the relation

$$\frac{z}{z^2 + (x - \xi)^2} = \frac{d}{d\xi}\tan^{-1}\frac{z}{x - \xi} \tag{3.14}$$

and integrating by parts. The resulting expression,

$$D_{2D}(x, z) = \frac{\theta}{2\pi} - \frac{i\nu}{2\pi}\int_{-\infty}^{0} d\xi\, e^{i\nu\xi}\tan^{-1}\frac{z}{x - \xi}, \tag{3.15}$$

is a point vortex at the origin with strength $\Gamma = e^{i\omega t}$, represented by the first term in (3.15), and a continuous vorticity distribution of strength

$$\gamma(\xi) = -i\nu\, e^{i\nu\xi + i\omega t}, \tag{3.16}$$

represented by the integral over the negative x-axis. Kelvin's theorem demands that the circulation around the point vortex and its wake is constant in time, or that the rate of change of circulation around the point vortex is convected into its wake with the inflow velocity $-U$,

$$i\omega\Gamma = -U\gamma|_{\xi=0}. \tag{3.17}$$

The validity of (3.17) is easy to confirm for $\Gamma = e^{i\omega t}$ and γ defined by (3.16).

The vortex velocity potential $D_{2D}(x, z)$ is the fundamental singularity in two-dimensional time-harmonic lifting problems. Its small-r expansion can be obtained by setting $k = 0$ in equation (3.11). It follows that (3.11) can be cast in the form

$$-\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}z}\tilde{S}(x, z; k) = D_{2D}(x, z) - \frac{1}{2\pi}z|k|L\left(\frac{v}{|k|}\right) + O(k^2r^2\ln r, v^2r^2\ln r). \tag{3.18}$$

Combining (3.18) with (3.5), and inverting back to the physical y-space, we obtain the desired inner expansion of the outer solution in the form

$$\phi(x, z; y) \sim \Gamma(y)\, D_{2D}(x, z) - \frac{z}{2\pi}\int_{-d}^{d}\Gamma'(\eta)K(y - \eta)\,\mathrm{d}\eta. \tag{3.19}$$

By virtue of the Fourier convolution theorem, the kernel $K(y)$ is defined by

$$K(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\, e^{-iky}\, i\,\mathrm{sgn}\,(k)\, L\left(\frac{v}{|k|}\right) = \tfrac{1}{2}\,\mathrm{sgn}\,(y)\left[\frac{e^{-v|y|}}{|y|} - ivE_1(v|y|) + vP(v|y|)\right], \tag{3.20}$$

where $E_1(x)$ is the exponential integral defined in Abramowitz and Stegun [1], and

$$P(y) = \int_1^{\infty}\mathrm{d}t\, e^{-yt}\left[\frac{(t^2 - 1)^{1/2} - t}{t}\right] + i\int_0^1\mathrm{d}t\, e^{-yt}\left[\frac{(1 - t^2)^{1/2} - 1}{t}\right]. \tag{3.21}$$

The complex analysis required for the derivation of (3.20)–(3.21) is outlined in Appendix 2.

The inner expansion of the outer solution (3.19), suggests the definition for the three-dimensional unsteady downwash $W(y)$,

$$W(y) = \frac{1}{2\pi}\int_{-d}^{d}\Gamma'(\eta)K(y - \eta)\,\mathrm{d}\eta, \tag{3.22}$$

where the function $K(y)$ is defined in (3.20)–(3.21). Its interpretation is the same as the three-dimensional downwash derived in the theory of Ahmadi and Widnall [2]. In particular, the two-dimensional downwash induced at each wing section by the wake of vorticity shed by the two-dimensional vortex of strength $\Gamma(y)$ corresponding to that section has been subtracted from $W(y)$ and is included in the leading-order term of the inner expansion

(3.19). This decomposition will permit the matching of the leading-order term in (3.19) to the strip-theory component of the inner solution and leads to the downwash W defined in (3.22) which tends to that in Prandtl's lifting-line theory in the zero-frequency limit and decays to zero at high frequencies of oscillation.

Expression (3.19) completes the two-term small-r expansion of the outer solution (3.1). The derivation of (3.19) placed no restriction on the wake wavelength relative to the wing span and thus is valid for all frequencies. The asymptotic order of the terms in the inner expansion of the outer solution (3.19) is analysed in Section 5. The circulation distribution $\Gamma(y)$ is unknown. Its magnitude will be determined by matching the inner expansion of the outer solution with an outer expansion of the inner solution, presented in Section 4.

4. Inner solution

At radial distances from the wing comparable to its chord, the gradual variation of its geometry and normal velocity suggest that the same is true for the flow gradients. The inner velocity potential satisfies the two-dimensional Laplace equation in the fluid domain, and on the mean position of the wing surface it is subject to the boundary condition (2.3) which is not affected by the high-aspect-ratio assumption. The inner problem is thus reduced to a sequence of two-dimensional time-harmonic lifting problems. A typical one is illustrated in Fig. 3, where the wavelength is drawn comparable to the chord for the sake of clarity.

The construction of the inner solution is facilitated by the definition of the normalized heave-velocity potential χ_3, subject to

$$\chi_{3z} = i\omega \tag{4.1}$$

on $z = \pm 0$ and $|x| < l$. The solution of the two-dimensional Laplace equation, supplemented by the boundary condition (4.1) and a Kutta condition at the trailing edge has been obtained for the general transient problem by Wu [18]. In the present special case of a time-harmonic motion and steady forward speed, two properties of χ_3 will be useful in the analysis of this section. Over distances r large compared to half the chord l, the flow can be approximated by a time-harmonic point vortex at the origin and its wake, or

$$\chi_3(x, z) = d_3(y)D_{2D}(x, z) + O(l/r, \nu l), \tag{4.2}$$

where the vortex velocity potential D_{2D} is defined by (3.15), and the circulation d_3 is given by the expression (Newman [9])

$$d_3(y) = 4U\, e^{-i\nu l}/[iH_0^{(2)}(\nu l) + H_1^{(2)}(\nu l)]. \tag{4.3}$$

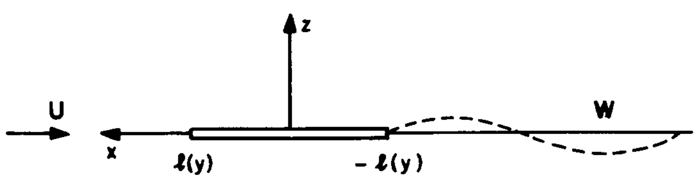

Fig. 3. Inner domain.

The complex lift coefficient associated with the velocity potential χ_3 is given by

$$C_{3L}^{2D} = \frac{L_3}{\frac{1}{2}\varrho U^2(2l)(i\omega/U)} = -2\pi C(\nu l) - \pi i \nu l, \tag{4.4}$$

where

$$C(x) = \frac{H_1^{(2)}(x)}{H_1^{(2)}(x) + iH_0^{(2)}(x)} \tag{4.5}$$

is the Theodorsen function, and $H_{0,1}^{(2)}$ the Hankel functions of the second kind defined in Abramowitz and Stegun [1]. The last term in (4.4) is the contribution from the added-mass force on the foil, taken to be a flat plate.

Similar results can be derived for the normalized pitch-velocity potential χ_5 which satisfies

$$\chi_{5z} = i\omega x \tag{4.6}$$

on the mean position of the foil. Its far-field expansion can be also written in the form

$$\chi_5(x, z) = d_5(y)D_{2D}(x, z) + O(l/r, \nu l) \tag{4.7}$$

with the same definition for D_{2D}, and

$$d_5(y) = -\tfrac{1}{2}ld_3(y), \tag{4.8}$$

where d_3 is defined in (4.3). The corresponding lift coefficient is

$$C_{5L}^{2D} = \frac{L_5}{\frac{1}{2}\varrho U^2(2l)(i\omega/U)} = \pi C(\nu l). \tag{4.9}$$

Analogous expressions can be also obtained for the moment coefficients associated with the velocity potentials χ_3 and χ_5.

The inner solution will be obtained next for the heaving and pitching motion as the sum of a particular and a homogeneous component. We set

$$\phi(x, z; y) = \phi_P(x, z; y) + F(y)\,\phi_H(x, z; y), \tag{4.10}$$

where ϕ_H is a normalized homogeneous solution, and $F(y)$ an "interaction" function which is a priori unknown. It depends parametrically on the y-coordinate and accounts for the hydrodynamic interactions in the spanwise direction. Its magnitude will be related to and determined together with the outer circulation distribution $\Gamma(y)$, following the asymptotic matching of the outer and inner solutions. This synthesis of the inner problem was used by Newman [10] for the radiation of surface waves by a slender ship, and by Ahmadi and Widnall [2] for this unsteady lifting problem.

For heave, a particular solution is

$$\phi_{3P} = \chi_3, \tag{4.11}$$

and for pitch

$$\phi_{5P} = \chi_5 - \frac{U}{i\omega}\chi_3. \tag{4.12}$$

The homogeneous solution ϕ_H is common to both heave and pitch, but the interaction functions $F_3(y)$ and $F_5(y)$ are different. The former is determined by enforcing the interaction of the foil with a vertical uniform flow. This construction is motivated by the downwash induced by the outer flow. Thus,

$$\phi_H(x, z) = i\omega z - \chi_3(x, z). \tag{4.13}$$

A more general homogeneous solution can be obtained by allowing the interaction of the foil with an oscillatory gust convected by the inflow velocity U, reducing to the solution of the Sears problem, as suggested by Ahmadi and Widnall [2]. In Section 5 it will be shown that the selection of the homogeneous solution (4.13) does not limit the uniform validity of the theory.

The inner solution for heave becomes

$$\phi_3 = \chi_3 + F_3(y)(i\omega z - \chi_3),$$

and for pitch

$$\phi_5 = \chi_5 - \frac{U}{i\omega}\chi_3 + F_5(y)(i\omega z - \chi_3). \tag{4.15}$$

Their outer expansion for $r \gg l$ is obtained by using (4.2) and (4.7):

$$\phi_3 \sim d_3(1 - F_3)D_{2D} + i\omega z F_3, \tag{4.16}$$

$$\phi_5 \sim \left[d_5 - d_3\left(F_5 + \frac{U}{i\omega}\right)\right] D_{2D} + i\omega z F_5. \tag{4.17}$$

Expressions (4.16) and (4.17) constitute a two-term outer expansion for the inner heave- and pitch-velocity potentials. In the next section they are matched with the corresponding expansion of the outer solution and an error analysis is carried out to establish the existence of an overlap region, as well as the frequency range over which the solution is uniformly valid.

5. Matching and error analysis

(a) Matching

The compatibility of the inner expansion of the outer solution (3.19) with the outer expansion of the inner solution (4.16)–(4.17) can be enforced by inspection. The matching conditions follow in the form

$$\Gamma_j(y) = \begin{cases} d_3(y)[1 - F_3(y)], & j = 3, \\ d_5(y) - d_3(y)\left[F_5(y) + \dfrac{U}{i\omega}\right], & j = 5, \end{cases} \tag{5.1}$$

$$-\frac{1}{2\pi}\int_{-d}^{d}\Gamma_j'(\eta)K(y-\eta)\,\mathrm{d}\eta = i\omega F_j(y), \quad j = 3, 5. \tag{5.2}$$

Eliminating the interaction functions $F_j, j = 3, 5$, we obtain an integro-differential equation for the outer distribution of circulation $\Gamma_j, j = 3, 5$, for heave and pitch,

$$\Gamma_j(y) - \frac{d_3(y)}{2\pi i\omega}\int_{-d}^{d}\Gamma_j'(\eta)K(y-\eta)\,\mathrm{d}\eta = \begin{cases} d_3(y), & j = 3, \\ d_5(y) - \dfrac{U}{i\omega}d_3(y), & j = 5, \end{cases} \tag{5.3}$$

where the kernel $K(y)$ is defined by (3.20).

The solution of (5.3) for Γ_j permits the determination of the interaction functions $F_j(y)$ from (5.2).

In the limit of zero frequency of oscillation, the kernel reduces to

$$K(y) = \frac{1}{2y} \tag{5.4}$$

and d_3, defined in (4.3), reduces to the steady-state two-dimensional circulation around a flat plate at a negative unit angle of attack and velocity $U = 1$,

$$d_3(y) = -2\pi l(y). \tag{5.5}$$

Substituting (5.4) and (5.5) in (5.3) for $j = 3$ we obtain

$$\Gamma(y) + \frac{l(y)}{2}\int_{-d}^{d}\frac{\Gamma'(\eta)}{y-\eta}\,\mathrm{d}\eta = -2\pi l(y), \tag{5.6}$$

which is the integral equation in Prandtl's lifting-line theory. In the opposite limit of large frequencies, the kernel becomes exponentially small, and

$$\Gamma_j(y) = \begin{cases} d_3(y), & j = 3, \\ d_5(y) - \dfrac{U}{i\omega}d_3(y), & j = 5, \end{cases} \tag{5.7}$$

which is the anticipated strip-theory solution.

The proper limiting behavior of the integral equation (5.3) at low and high frequencies suggests its uniform validity in the intermediate frequency range.

(b) Error analysis

The objective is to show that there exists an "intermediate" region in which the relative errors in the inner and outer expansions of the outer and inner solutions respectively are small in the limit of large aspect ratio. Moreover, the wake-wavelength range will be established over which the determination of such an intermediate region is possible.

Errors in the inner expansion of the outer solution originate from the small-r approximation (3.11) of the ascending-series expansion (3.9). The ratio of the terms neglected to the leading-order term in (3.11) is of order

$$\mathrm{O}(k^2r^2,\ v^2r^2), \tag{5.8}$$

where vd is the reduced frequency. It has been assumed that $\ln r$ terms and the Fourier wavenumber k are of O(1).

The corresponding errors in the outer expansion of the inner solution originate from three sources. The approximation of the Laplace equation involves an error of $\mathrm{O}(k^2r^2)$. This follows from the observation that the two-dimensional Laplace equation is the $k = 0$ limit of the y-Fourier transform of its three-dimensional form, which is the modified Helmholtz equation defined in (3.7). An error of order $\mathrm{O}(1/Ar,\ vl)$ is present in the approximation of the particular solution by a point vortex, and an error of $\mathrm{O}(v^2r^2)$ in the selection of the homogeneous solution (4.13). Summarizing, errors of order

$$\mathrm{O}\left(k^2r^2,\ v^2r^2,\ vl,\ \frac{1}{Ar}\right) \tag{5.9}$$

are present in (4.16)–(4.17).

Let $\varepsilon = 1/A$ be the inverse of the aspect ratio. The intermediate region is located at a distance $r = \mathrm{O}(\varepsilon^\alpha)$ from the wing axis, where

$$0 < \alpha < 1. \tag{5.10}$$

The upper limit $\alpha = 1$ defines the inner, and the lower limit $\alpha = 0$ the outer regions, respectively. Also let the wake wavelength to be of order

$$\lambda = \frac{2\pi}{v} = \mathrm{O}(\varepsilon^\gamma) \tag{5.11}$$

where $\gamma = 0$ corresponds to wavelengths comparable to the wing span, and $\gamma = 1$ to wavelengths comparable to its chord. In terms of ε, the errors (5.8) and (5.9) can be expressed in the form

$$\mathrm{O}(\varepsilon^{2\alpha},\ \varepsilon^{\alpha-\gamma},\ \varepsilon^{2\alpha-2\gamma},\ \varepsilon^{1-\alpha}). \tag{5.12}$$

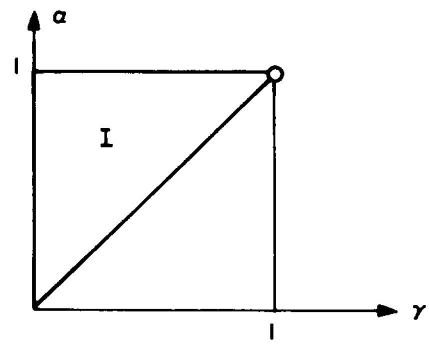

Fig. 4. Diagram of asymptotic validity of the theory.

For all terms in (5.12) to be small in the limit $\varepsilon \to 0$, the following inequalities must hold

$$\alpha > 0, \quad \alpha > \gamma, \quad \alpha < 1. \tag{5.13}$$

Domain I in Fig. 4 illustrates the domain of validity of (5.13) in the α, γ plane, and indicates that for any $0 < \gamma < 1$ there exists an intermediate region of finite α-extent in which the relative errors of the outer and inner solutions are small in the limit $\varepsilon \to 0$. Consequently, the asymptotic analysis is valid for $0 < \gamma < 1$, with the value $\gamma = 1$ excluded. The proper limiting behavior, for wavelengths comparable to the wing chord, suggests the usefulness of the theory in the extended frequency range. It will be hereafter referred to as unified theory.

6. Order-of-magnitude analysis

In the analysis of the present section, finite-aspect-ratio corrections are understood to apply to the ratio Γ_{3D}/Γ_{2D} as $A \to \infty$.

The selection of the appropriate gauge functions in the present problem is suggested by the inner expansions of the outer solution, (3.11) and (3.18). Combining (3.11) with (3.5) and inverting back to the physical y-space, we obtain

$$\phi(x, z; y) = \frac{\Gamma(y)}{2\pi}\left\{\theta - \frac{\pi v z}{2} - \frac{iv}{2}[-2z(\gamma - 1 + \ln vr) - 2\theta x]\right\}$$

$$- \frac{z}{2\pi}\int_{-d}^{d} \mathrm{d}\eta \Gamma'(\eta) K(y - \eta) + \mathrm{O}(v^2 r^2 \ln r) \tag{6.1}$$

where the kernel $K(y)$ is defined in (3.20)–(3.21). In terms of the inner variables

$$X = Ax, \tag{6.2}$$

$$Z = Az, \tag{6.3}$$

the sequence of gauge functions which are relevant in the inner expansion (5.1) depends on the order of magnitude of the reduced frequency νd. The appropriate sequence of gauge functions is considered in three asymptotic frequency regimes:

i) $\nu = O(1/A)$: very low frequencies,

$$1, \frac{1}{A}, \frac{1}{A^2} \ln A, \ldots, \tag{6.4}$$

(ii) $\nu = O(1)$: low frequencies,

$$1, \frac{1}{A} \ln A, \frac{1}{A}, \ldots, \tag{6.5}$$

iii) $\nu = O(A)$: high frequencies.

In this high-frequency case (6.1) is an inappropriate inner expansion because $\nu r = O(1)$ as $A \to \infty$.

Comparing cases i) and ii), we may conclude that the use of (6.1) leads to a non-uniformity in the choice of gauge functions, depending on the assumption on the order of the reduced frequency. This non-uniformity can be removed if the small-r expansion of the oscillatory two-dimensional point vortex defined in (3.15) is introduced,

$$D_{2D}(y, z) = \frac{\theta}{2\pi} - \frac{\nu z}{4} - \frac{i\nu}{4\pi}\left[(x^2 - z^2)\frac{\sin\theta}{r} - 2z(\ln \nu r - 1) - 2\theta x - \frac{2xz\cos\theta}{r}\right]$$
$$+ O(\nu^2 r^2 \ln r). \tag{6.6}$$

Comparing (6.1) and (6.6), we are led to a uniform inner expansion of the outer solution given by equation (3.19), reproduced here

$$\phi(x, z; y) = \Gamma(y)\, D_{2D}(y, z) - \frac{z}{2\pi}\int_{-d}^{d} d\eta \Gamma'(\eta) K(y - \eta) + O(\nu^2 r^2 \ln r), \tag{6.7}$$

with a relative error comparable to that in (6.1), and of order $(1/A) \ln A$. The function D_{2D} in (6.7) unifies the three frequency-dependent gauge-function sequence

$$g_0(A) = O[F(\nu, A)],\ g_1(A) = O\left[\frac{1}{A} f(y, \nu)\right], \ldots, \tag{6.8}$$

where the functions $F(A, \nu)$ and $f(y, \nu)$ are defined by

$$F(\nu A) = \frac{\theta}{2\pi} - \frac{i\nu A}{2\pi}\int_{-\infty}^{0} d\,\Xi\, e^{i\nu A\Xi} \tan^{-1}\frac{z}{x - z}, \tag{6.9}$$

$$f(y, \nu) = \int_{-d}^{d} d\eta \Gamma'_{2D}(\eta) K(y - \eta), \tag{6.10}$$

where $X, Z = O(1)$ and $O(\Gamma_{2D}) = 1$ since we are interested in the order of magnitude of the finite-aspect-ratio correction relative to the two-dimensional solution. For all cases (i)–(iii), it may be verified that

$$g_0(A) = 1. \tag{6.11}$$

For $\nu = O(1/A)$ or $\nu = O(1)$, (cases (i) and (ii)),

$$g_0(A) = 1, g_1(A) = \frac{1}{A}, \ldots, \tag{6.12}$$

and for $\nu = O(A)$,

$$g_0(A) = 1, g_1(A) = \frac{1}{A^3}, \ldots. \tag{6.13}$$

The last result follows from the large-y asymptotic behavior of the function $L(y)$ defined in (3.12), ($L(y) \sim 1/y^2$, as $y \to \infty$), and justifies the validity of strip theory to leading-order for high frequencies.

The sequence (6.13) appears to imply that for high frequencies the magnitude of the high-aspect-ratio corrections is of order $1/A^3$ which appears not to be consistent with the theory of Guiraud and Slama [5], where it is shown that the three-dimensional effects are of $O(A^{-2} \ln A)$. The present theory accounts for finite-aspect-ratio corrections of order A^{-1} relative to the leading-order two-dimensional solution. Since the three-dimensional Laplace equation is approximated by its two-dimensional form in the inner problem, terms of $O(A^{-2})$ are dropped in the inner expansion of the outer solution. Consequently, at high-frequencies, strip theory is recovered to leading order but finite-aspect-ratio corrections are not accounted for since their order is comparable to the asymptotic error in the present theory. Thus the results in the present paper are uniformly valid in the frequency domains I–IV defined in the Introduction, up to and including finite-aspect-ratio corrections of $O(1/A)$ relative to the leading-order two-dimensional solution.

Summarizing, a sequence of gauge functions uniformly valid in the frequency range $\nu d < O(A)$, is

$$1, g_1(A), \ldots, \tag{6.14}$$

with the function $f(y, \nu)$ defined in (6.10).

The sequence (6.14) suggests the order of magnitude of the particular and homogeneous components in the inner solution. It follows that the interaction functions $F_j(y), j = 3, 5$, are of order $g_1(A)$ relative to the particular and normalized homogeneous solutions.

Let

$$\Gamma_j = \Gamma_j^{(0)} + g_1(A)\Gamma_j^{(1)} + \ldots, \tag{6.15}$$

$$F_j = F_j^{(0)} + g_1(A)F_j^{(1)} + \ldots. \tag{6.16}$$

A term-by-term matching can be carried out by substituting the expansions (6.15) and (6.16) in (5.1) and (5.2), and by equating terms of O(1) and O[$g_1(A)$]. This procedure, here applied for heave, leads to:

O(1) matching:

$$\Gamma_3^{(0)} = d_3(y), \tag{6.17}$$

$$F_3^{(0)} = 0. \tag{6.18}$$

O[$g_1(A)$] matching:

$$\Gamma_3^{(1)} = -d_3(y)F_3^{(1)}, \tag{6.19}$$

$$F_3^{(1)} = -\frac{1}{2\pi i\omega}\int_{-d}^{d} \mathrm{d}\eta\,\Gamma_3^{(0)\prime}(\eta)K(y-\eta), \tag{6.20}$$

and corresponds to the analysis of Van Dyke [16] for the steady-state problem. Due to the success of an analogous equation in the ship radiation problem, the direct numerical solution

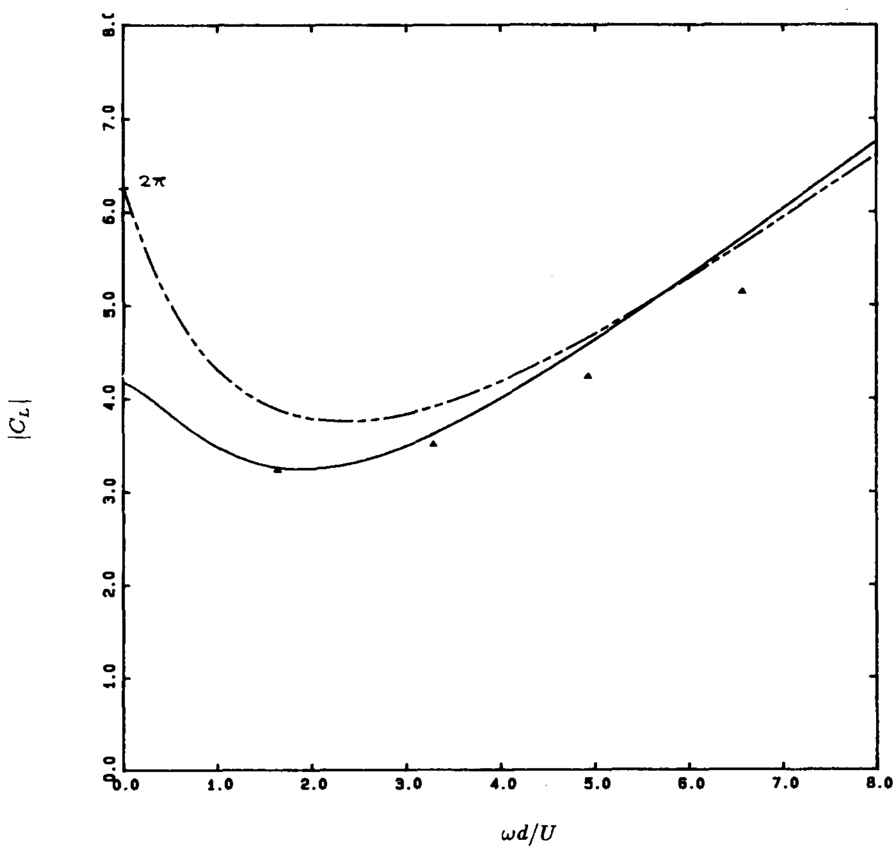

Fig. 5. Modulus of the heave lift-coefficient defined in (7.2) for a wing of elliptical planform of aspect ratio $A = 4$, as a function of the reduced frequency made non-dimensional by the wing span. (- - - -) Strip theory, (——) Unified theory, (▲) Numerical solution of Lee [8] using a vortex-lattice method.

of the integral equation (5.3) is preferred here. The computational overhead is associated with the solution of a complex linear system with unknowns the number of terms in the Fourier series necessary to approximate the distribution of circulation along the span. This effort is small compared to the evaluation of the quadratures which involve the kernel $K(y)$ in the integral equation or in (6.20).

7. Numerical solution

The numerical solution of the integral equation has been carried out for two planforms symmetric with respect to the x-axis. The circulation $\Gamma(y)$ has been expanded in the sine Fourier series

$$\Gamma(y) = \sum_{k=0}^{N} \Gamma_{2k+1} \sin(2k+1)\theta, \quad y = d\cos\theta, \tag{7.1}$$

and upon substitution in (5.3), the discretized integral equation is satisfied at a set of collocation points selected to be the local maxima of $\sin(N+1)\theta$. The product of $\sin(2k+1)\theta$ with the singular part of the kernel was integrated analytically over the span. Glauert's integrals were used for the integration of the Cauchy singularity. The remaining regular parts were integrated by using Romberg's quadrature to single-precision accuracy. The resulting linear system was solved by direct Gauss reduction. In all computations reported here, $N = 8$.

The lift-coefficient modulus and phase was evaluated for two wings of aspect ratio $A = 4$ forced in a time-harmonic heave oscillation. The first has an elliptical and the second a

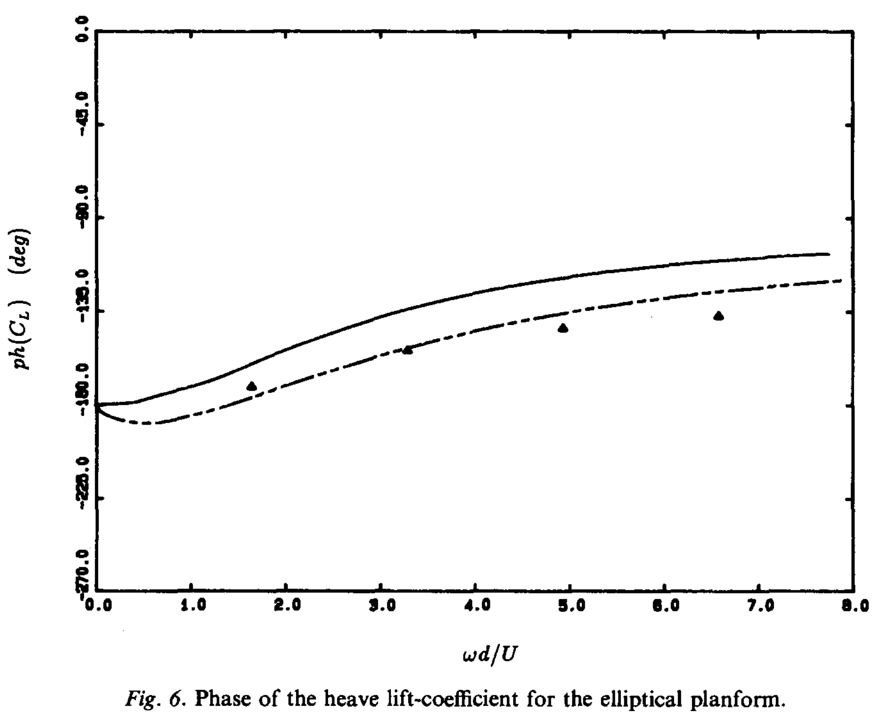

Fig. 6. Phase of the heave lift-coefficient for the elliptical planform.

rectangular planform. Formally, the theory is not expected to be valid near the tips of lifting surfaces with non-vanishing chord, due to the local three-dimensional effects not accounted for by the present asymptotic analysis. In the numerical scheme no collocation points are located at the tips, and its implementation for rectangular planforms is in principle possible. The complex heave lift-coefficient can be obtained by combining (4.4), (4.10) and integrating along the wing span. The result is

$$C_L = \frac{L_3}{1/2\varrho U^2 Siv\xi_3} = -\frac{4}{S}\int_{-d}^{d} \mathrm{d}y C(vl)\, l(y) F_3(y) - iv\frac{a_{33}}{S}, \tag{7.2}$$

where the interaction function $F_3(y)$ is determined from (6.2) following the solution of (6.3), and a_{33} is the heave added-mass of the planform area.

In (7.2) the first term is the contribution to the vertical force due to circulatory effects, and the last is the added-mass contribution. The latter is proportional to the vertical acceleration of the wing surface and vanishes at low frequencies. At the same limit, the lift tends to the prediction of Prandtl's lifting-line theory. In the opposite limit of high frequencies, the lift tends to a constant value, while the added-mass force grows like the square of the frequency of oscillation. For the elliptical planform, the added-mass coefficient was evaluated exactly

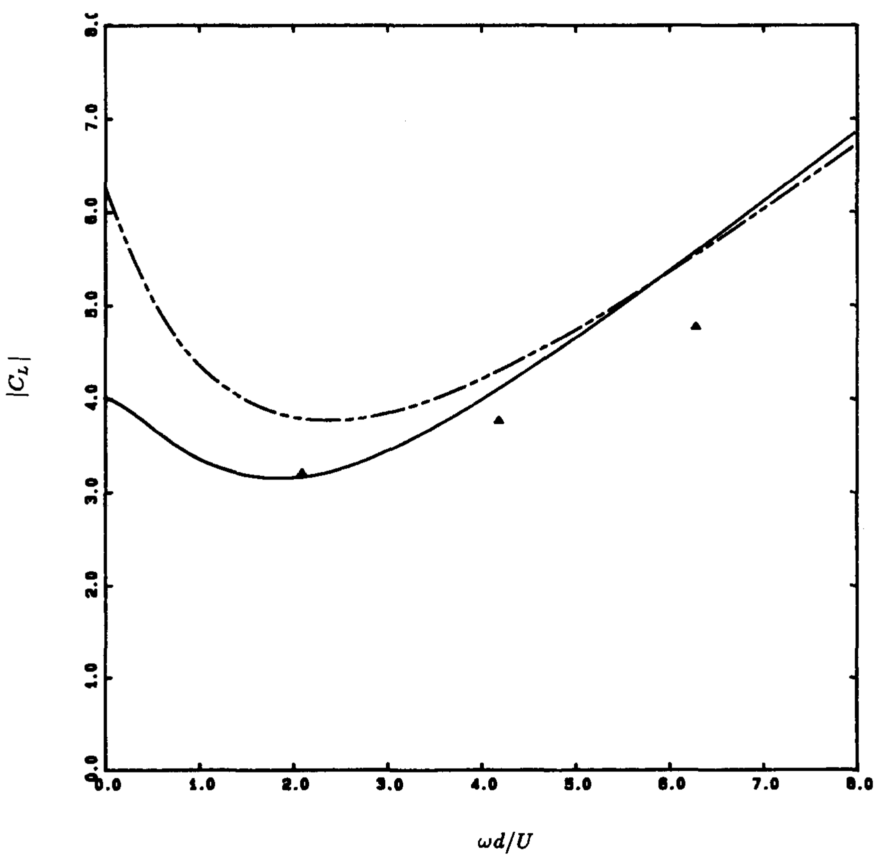

Fig. 7. Modulus of the heave lift-coefficient for a wing with rectangular planform and aspect ratio $A = 4$.

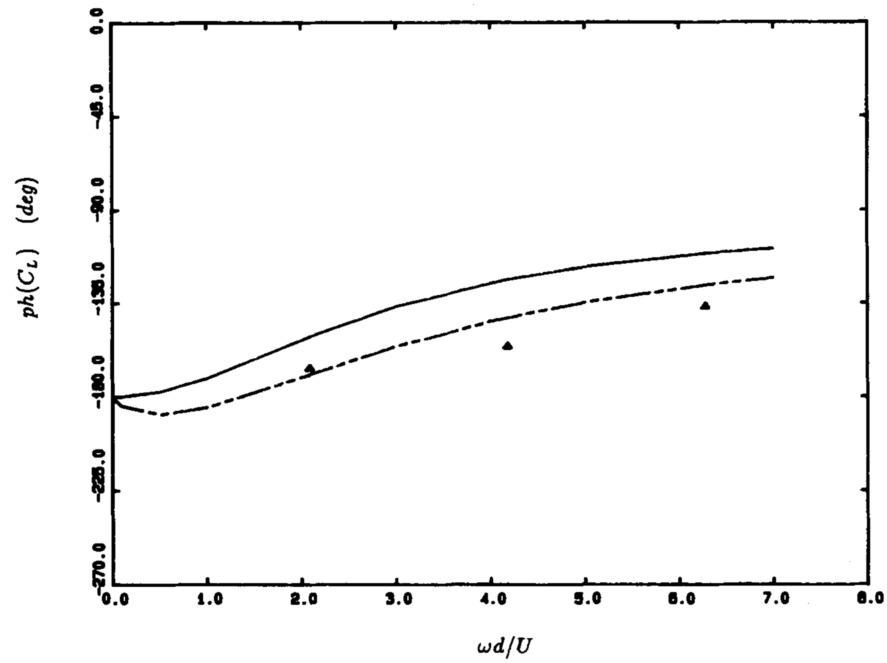

Fig. 8. Phase of the heave lift-coefficient for the rectangular planform.

from the closed-form expression derived in Lamb [7]. Strip theory has been used for the rectangular planform.A numerical solution of the three-dimensional transient lifting-surface problem for the two planforms has been carried out by Lee [8] using the vortex-lattice method. In Lee's solution the wing time-harmonic oscillation starts from rest, and transients are filtered out after a few periods by a harmonic analysis. Eight vortex elements were used along the chord and four along half the span. For the elliptical planform, the modulus of the heave lift-coefficient predicted by strip theory, unified theory and the lifting-surface solution are presented in Fig. 5. The lift-phase difference from the vertical displacement of the wing surface is illustrated in Fig. 6. The corresponding results for the rectangular planform are shown in Figs. 7 and 8. The agreement between unified theory and the three-dimensional solution is very good at moderate and low frequencies, but less so at high frequencies. For the elliptical wing, the unified-theory high-frequency limit for the modulus of the lift coefficient is correct since it is dominated by the added-mass contribution (last term in (7.2)) which is known exactly for an elliptical plate. In the zero-frequency limit, unified theory reduces to Prandtl's lifting-line theory, while for high frequencies, strip and unified theory converge to the same asymptotic limit.

8. Conclusions

An unsteady lifting-line theory has been developed for unswept wings of high aspect-ratio forced to oscillate in a time-harmonic manner. It generalizes existing studies in that it is uniformly valid in the reduced-frequency range $0 < \nu d < O(A)$. The method of matched asymptotic expansions is used. In an outer domain the solution is expressed in terms of a distribution of normal dipoles over the wake with moment varying sinusoidally in the streamwise direction. A two-term expansion near the wing axis consists of a leading-order term equal to the velocity potential due to a two-dimensional oscillatory point vortex. The next-order correction is the oscillatory downwash. Near the wing surface the flow is approximated in a two-dimensional manner. The particular strip-theory solution is supplemented by a homogeneous component which accounts for the finite-aspect-ratio correction induced by the downwash. The matching of the outer and inner solutions produces an integral equation for the circulation distribution along the wing axis. In the zero-frequency limit it reduces to Prandtl's lifting-line equation and for high frequencies to strip theory. Comparisons with lifting-surface numerical calculations of the lift of two wings of aspect ratio 4 are very satisfactory.

A study of the gauge functions in the asymptotic expansion for the spanwise circulation distribution revealed that their dependence on the aspect ratio A is non-uniform. It depends on the assumed order of magnitude for the reduced frequency. The present theory suggests that they are frequency-dependent and expressions for them are derived. The gauge functions for very low, low and high frequencies are obtained as special cases.

The extension of the present theory to the evaluation of the unsteady force and moment due to an incident sinusoidal gust is straightforward. Moreover, its uniform validity over a wide frequency range, extends its usefulness to transient problems where the Fourier transform of the frequency domain forces can be utilized.

Acknowledgements

The author wishes to thank Professors J. Nicholas Newman and Patrick Leehey of MIT for useful discussions on the subject. He also expresses his appreciation for the financial support provided by the National Science Foundation (Grant 8210649-A01-MEA) and the Office of Naval Research (Special Focus Program on Numerical Ship Hydrodynamics).

Appendix 1: Ascending-series expansion of outer solution

For $z = 0$ and $x > 0$ the function $\tilde{S}(x, 0; k)$ takes the form

$$\tilde{S}(x, 0; k) = 2 \int_0^\infty d\xi\, e^{-i\nu\xi} K_0[|k|(x + \xi)]. \tag{A1.1}$$

Using the addition theorem (3.8) in (A1.1), and interchanging the order of integration, we obtain

$$\tilde{S}(r, 0; k) = \frac{4}{\pi} \int_0^\infty d\tau\, K_{i\tau}(|k|r) G(\tau), \tag{A1.2}$$

where

$$G(\tau) = \int_0^\infty d\xi\, e^{-i\nu\xi} K_{i\tau}(|k|\xi). \tag{A1.3}$$

Using the integral definition of the modified Bessel function $K_{i\tau}(x)$,

$$K_{i\tau}(z) = \int_0^\infty dt \cos(\tau t)\, e^{-z} \cosh t, \tag{A1.4}$$

in (A1.3), and interchanging the order of integration we obtain

$$G(\tau) = \int_0^\infty dt \cos(\tau t) \int_0^\infty e^{-\xi(|k|\cosh t + iv)} d\xi = \frac{1}{|k|}\int_0^\infty dt \frac{\cos(\tau t)}{\cosh t + iv/|k|} = \frac{\pi}{|k|}\frac{\sin \alpha\tau}{\sinh \alpha \sinh \pi\tau}, \tag{A1.5}$$

where $\cosh \alpha = iv/|k|$. Substituting (A1.5) in (A1.2), using the integral definition (A1.4) and the identity

$$\int_0^\infty d\mu \cos(\mu s) \frac{\sin \alpha\mu}{\sinh \pi\mu} = \frac{1}{2}\frac{\sinh \alpha}{\cosh s + \cosh \alpha}, \tag{A1.6}$$

we obtain

$$\tilde{S}(r, 0; k) = \frac{2}{|k|}\int_0^\infty d\mu \frac{e^{-|k| r \cosh \mu}}{\cosh \mu - \cosh \gamma}, \tag{A1.7}$$

where $\cosh\gamma = -iv/|k|$. The small-r expansion of (A1.7) is obtained by employing a technique used by Ursell [15] in the problem of surface-wave radiation by a slender body. The Laplace transform of (A1.7) is first reduced to the form

$$\int_0^\infty dr\, e^{-|k| r \cosh \beta} \tilde{S}(r) = \frac{2}{k^2}\int_0^\infty d\mu \frac{1}{\cosh \mu - \cosh \gamma} \cdot \frac{1}{\cosh \mu + \cosh \beta}$$

$$= \frac{2}{k^2}\frac{1}{\cosh \beta + \cosh \gamma}\left\{\int_0^\infty \frac{d\mu}{\cosh \mu - \cosh \gamma} - \int_0^\infty \frac{d\mu}{\cosh \mu + \cosh \beta}\right\}, \tag{A1.8}$$

and use is made of the integral identity

$$\int_0^\infty \frac{d\mu}{\cosh \mu + \cosh c} = \frac{c}{\sinh c} \tag{A1.9}$$

to obtain

$$\mathscr{L}(\tilde{S}) = \frac{2}{k^2}\frac{1}{\cosh \beta + \cosh \gamma}\left(\frac{\gamma^*}{\sinh \gamma^*} - \frac{\beta}{\sinh \beta}\right). \tag{A1.10}$$

Expanding the fractions of hyperbolic sines and cosines in (A1.10) in the series

$$\frac{\sinh \beta}{\cosh \beta + \cosh \gamma} = 1 + 2\sum_{m=1}^\infty (-)^m \cosh m\gamma\, e^{-m\beta}, \tag{A1.11}$$

$$\frac{1}{\cosh \beta + \cosh \gamma} = \frac{2}{\sinh \gamma}\sum_{m=1}^\infty (-)^{m-1} \sinh m\gamma\, e^{-m\beta}, \tag{A1.12}$$

and using the identities

$$\int_0^\infty dr\, e^{-|k| r \cosh \beta} I_v(|k|r) = e^{-v\beta}/|k| \sinh \beta, \tag{A1.13}$$

$$\int_0^\infty dr\, e^{-|k| r \cosh \beta} \frac{\partial I_v(|k|r)}{\partial v} = -\frac{\beta\, e^{-v\beta}}{|k| \sinh \beta} \tag{A1.14}$$

to invert the Laplace transform (A1.7), we obtain the desired ascending series

$$\tilde{S}(r, 0; k) = \frac{2}{|k|}\left\{\frac{\gamma^*}{\sinh \gamma^*}\left[I_0(|k|r) + 2\sum_{m=1}^\infty (-)^m \cosh m\gamma\, I_m(|k|r)\right]\right.$$

$$\left. + \frac{2}{\sinh \gamma}\sum_{m=1}^\infty (-)^{m-1} \sinh m\gamma \left[\frac{\partial I_v(|k|r)}{\partial v}\right]_{v=m}\right\}, \tag{A1.15}$$

valid for $z = 0$ and $x > 0$. To obtain the ascending series of $\tilde{S}$ for $z \neq 0$, we observe that $\tilde{S}$ is an even function of z and satisfies the modified Helmholtz equation (3.7). Thus, its values on the $z = 0$ axis define the function in the whole x, z plane. Its analytic continuation is

$$\tilde{S}(r, 0; k) = \frac{2}{|k|}\left\{\frac{\gamma^*}{\sinh \gamma^*}\left[I_0(|k|r) + 2\sum_{m=1}^{\infty} (-)^m \cosh m\gamma\, I_m(|k|r) \cos m\theta\right]\right.$$

$$\left. + \frac{2}{\sinh \gamma}\sum_{m=1}^{\infty} (-)^{m-1} \sinh m\gamma \frac{\partial}{\partial v}[I_v(|k|r)\cos v\theta]_{v=m}\right\}. \quad \text{(A1.16)}$$

Expression (A1.16) is the desired ascending-series expansion, valid generally for non-zero values of r and θ. At $\theta = \pi$ it possesses a jump discontinuity caused by the derivative $\partial(\cos v\theta)/\partial v|_{v=m}$.

Appendix 2: Reduction of the kernel

The Fourier integral (3.20) is an odd function of y. This property follows if we apply the transformation $k = -k'$, and use the symmetry of $L(v/|k|)$ with respect to k. The reduction of (3.20) will be carried out by contour integration in the complex k-plane using Cauchy's integral theorem. It is hereafter assumed that $x > 0$.

As $k \to \infty$, $L(v/|k|) \to \pi/2$. Rewrite

$$L(v/|k|) = \frac{\pi}{2} + W(v/|k|). \quad \text{(A2.1)}$$

Substituting the first term of (A2.1) in (3.20) we obtain

$$\frac{i}{4}\int_{-\infty}^{\infty} dk\, e^{-iky} \operatorname{sgn} k = \frac{1}{4y}\int_{-\infty}^{\infty} dk\, e^{-iky}\, 2\delta(k) = \frac{1}{2y}. \quad \text{(A2.2)}$$

The remaining integral over the positive and negative real k-axis can be written as the sum of two integrals

$$K_R = K_1 + K_2, \quad \text{(A2.3)}$$

$$K_1(\xi) = -\frac{iv}{2\pi}\int_0^{\infty} dk\, e^{ik\xi}\, W(1/k), \quad \text{(A2.4)}$$

$$k_2(\xi) = \frac{iv}{2\pi}\int_0^{\infty} dk\, e^{-ik\xi}\, W(1/k), \quad \text{(A2.5)}$$

where $\xi = vy > 0$.

The complex function $W(1/k)$ is now defined as follows

$$W(1/k) = \frac{\cosh(i/k)}{[(i/k)^2 - 1]^{1/2}}(1 + 1/k^2) + \frac{i}{k}\ln\frac{2}{k} - \frac{\pi}{2k} - \frac{\pi}{2}. \quad \text{(A2.6)}$$

It vanishes as $k \to \infty$, and is cut in the complex k-plane along the negative imaginary axis.

Define the transformation

$$w = \frac{i}{k} = |W|\, e^{i\phi}, \quad k = r\, e^{i\theta}, \quad \text{(A2.7)}$$

and deviate the contour of integration for the integrals K_1 and K_2 in the upper and lower k-halfplanes respectively.

Along $k = ir$, $\theta = \pi/2$ and $w = 1/r$, $\phi = 0$:

$$0 < r < 1: \; W = \frac{\cosh^{-1}(1/r)}{[(1/r)^2 - 1]^{1/2}}(1 - 1/r^2) + \frac{1}{r}\ln\frac{2}{r} - \frac{\pi}{2},$$

$$1 < r < \infty: \; W = \frac{\cos^{-1}(1/r)}{[(1 - (1/r)^2]^{1/2}}(1 - 1/r^2) + \frac{1}{r}\ln\frac{2}{r} - \frac{\pi}{2}.$$

Along $k = -ir$, $\theta = \pi/2$, and $w = -1/r$, $\phi = \pi$:

$$0 < r < 1: W = -\frac{\cosh^{-1}(1/r)}{[(1/r)^2 - 1]^{1/2}}(1 - 1/r^2) - \frac{1}{r}\ln\frac{2}{r}$$
$$+ \pi i\,\{[(1/r)^2 - 1]^{1/2} - 1\} - \frac{\pi}{2},$$

$$1 < r < \infty: W = \frac{\cos^{-1}(-1/r)}{[1 - (1/r)^2]^{1/2}}(1 - 1/r^2) - \frac{1}{r}\ln\frac{2}{r}$$
$$-\frac{\pi i}{2r} - \frac{\pi i}{2r} - \frac{\pi}{2}.$$

Using the above definitions in the integrals K_1 and K_2 we obtain for their sum;

$$K_R(\xi) = \frac{\nu}{2}\left\{\int_1^\infty \mathrm{d}r\, \mathrm{e}^{-\xi r}\left[\frac{(r^2 - 1)^{1/2}}{r} - \frac{i}{r} - 1\right] + \int_0^1 \mathrm{d}r\, \mathrm{e}^{-\xi r}\left[i\frac{(1 - r^2)^{1/2}}{r} - \frac{i}{r} - 1\right]\right\} \tag{A2.8}$$

In the process, the inverse cosines and logarithms cancelled out. Using the definition of the exponential integral

$$E_1(x) = \int_1^\infty \mathrm{d}r\, \frac{\mathrm{e}^{-xr}}{r}, \tag{A2.9}$$

and

$$\int_0^1 \mathrm{d}r\, \mathrm{e}^{-\xi r} = \frac{1 - \mathrm{e}^{-\xi}}{\xi} \tag{A2.10}$$

in (A2.8), and combining the result with (A2.2), we obtain the reduced form for the kernel

$$K(y) = \tfrac{1}{2}\operatorname{sgn}(y)\left\{\frac{\mathrm{e}^{-\nu|y|}}{|y|} - i\nu E_1(\nu|y|) + \nu P(\nu|y|)\right\}, \tag{A2.11}$$

where $P(y)$ is defined in (3.21).

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